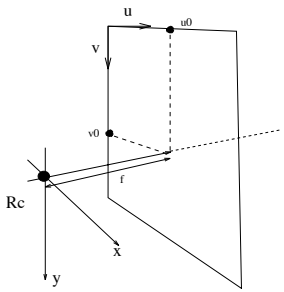
Geometry for Vision

2014



Content

1 Before beginning
2 Camera models
3 3D vision

Content

1 Before beginning
1 Homogeneous Coordinates
2 Homogeneous Transformations
2 Camera models
3 3D vision

Homogeneous Coordinates

Introduction:

Two parallel lines can intercept?

In Euclidean space (geometry), two parallel lines on the same plane cannot intercept

In projective space, the train railroad on the side picture becomes narrower while it moves far away from eyes. Finally, the two parallel rails meet at the horizon, which is a point at infinity

Solution: Homogeneous Coordinates

Homogeneous — Cartesian

$$(1,2,3) = (2,4,6) = (1a, 2a, 3a) \rightarrow \left(\frac{1}{3}, \frac{2}{3}\right)$$

Solution: Homogeneous Coordinates

Homogeneous coordinates are a way of representing N-dimensional coordinates with N+1 numbers

Therefore, a point in Cartesian coordinates, (X, Y) becomes (x, y, w) in Homogeneous coordinates

Homogeneous — Cartesian

$$(x, y, w) \leftrightarrow \left(\frac{x}{w}, \frac{y}{w}\right)$$

Solution:

Two parallel lines can intercept?

$$A\frac{x}{w} + B\frac{y}{w} + C = 0$$
$$A\frac{x}{w} + B\frac{y}{w} + D = 0$$

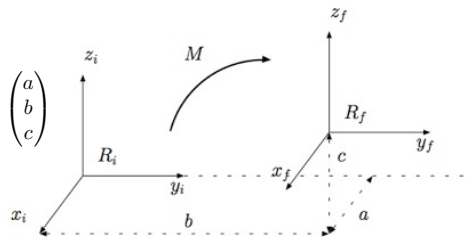
$$Ax + By + Cw = 0$$
$$Ax + By + Dw = 0$$

$$\text{Solution: } w = 0$$
$$(x, y, 0) \text{ point at infinity}$$

Introduction:

Coding rotation and translation in the same matrix

$$\begin{pmatrix} X^{R_f} \\ Y^{R_f} \\ Z^{R_f} \end{pmatrix} = \begin{pmatrix} X^{R_i} \\ Y^{R_i} \\ Z^{R_i} \end{pmatrix} - \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$



Translation

Introduction:

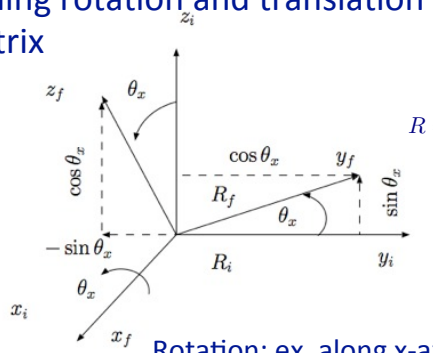
Coding rotation and translation in the same matrix

$$\begin{pmatrix} X^{R_f} \\ Y^{R_f} \\ Z^{R_f} \end{pmatrix} = \mathtt{R} \begin{pmatrix} X^{R_i} \\ Y^{R_i} \\ Z^{R_i} \end{pmatrix} - \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Combining rotation and translation

Introduction:

Coding rotation and translation in the same matrix

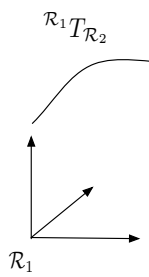


$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{pmatrix}$$

$$\begin{pmatrix} X^{R_f} \\ Y^{R_f} \\ Z^{R_f} \end{pmatrix} = \mathtt{R} \begin{pmatrix} X^{R_i} \\ Y^{R_i} \\ Z^{R_i} \end{pmatrix}$$

Rotation: ex. along x-axis

Homogeneous Transformations



$$\mathbf{P}^{R_1} \propto {}^{\mathcal{R}_1}T_{\mathcal{R}_2} \cdot \mathbf{P}^{R_2}$$

Rotation Translation

$$\begin{pmatrix} X^{R_1} \\ Y^{R_1} \\ Z^{R_1} \\ 1 \end{pmatrix} \propto \begin{pmatrix} s_x & n_x & a_x & P_x \\ s_y & n_y & a_y & P_y \\ s_z & n_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} X^{R_2} \\ Y^{R_2} \\ Z^{R_2} \\ 1 \end{pmatrix}$$

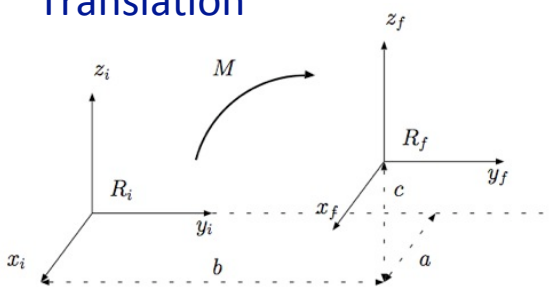
Into projective space

Cartesian Homogeneous

$$P = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} ax \\ ay \\ az \\ a \end{pmatrix}$$

$$\begin{pmatrix} X^{R_f} \\ Y^{R_f} \\ Z^{R_f} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -a \\ 0 & \cos\theta_x & -\sin\theta_x & -b \\ 0 & \sin\theta_x & \cos\theta_x & -c \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} X^{R_i} \\ Y^{R_i} \\ Z^{R_i} \\ 1 \end{pmatrix}$$

Translation



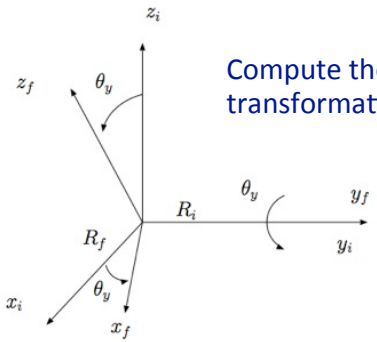
$$M = \begin{pmatrix} & & & a \\ & I_3 & & b \\ & & & c \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rigid Transformations

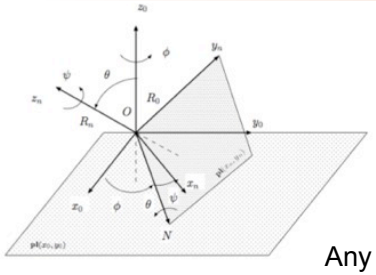
$$\begin{pmatrix} X^{R_1} \\ Y^{R_1} \\ Z^{R_1} \\ 1 \end{pmatrix} \propto \begin{pmatrix} s_x & n_x & a_x & P_x \\ s_y & n_y & a_y & P_y \\ s_z & n_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} X^{R_2} \\ Y^{R_2} \\ Z^{R_2} \\ 1 \end{pmatrix}$$

$$\mathbf{P}^{R_1} = \begin{pmatrix} {}^{R_1}\mathbf{R}_{R_2} & {}^{R_1}\mathbf{T}_{R_2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X^{R_2} \\ Y^{R_2} \\ Z^{R_2} \\ 1 \end{pmatrix} = \begin{pmatrix} X^{R_1} \\ Y^{R_1} \\ Z^{R_1} \\ 1 \end{pmatrix}$$

Exercice: y-axis rotation



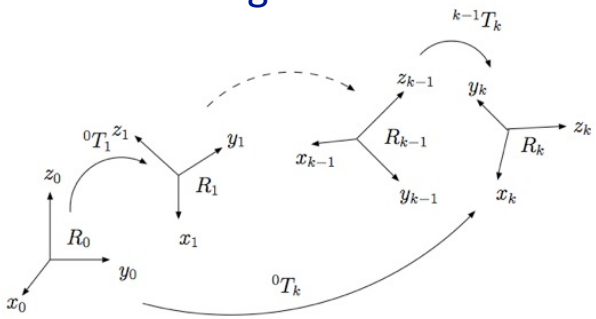
Compute the homogeneous transformation matrix



Any rotation can be written with three simple rotations:

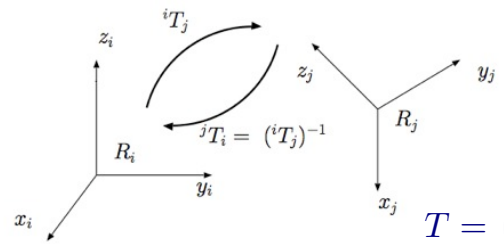
- Euler Angles,
- Quaternion,
...

Combining transformations



$$^0T_k = {}^0T_1.{}^1T_2.{}^2T_3.{}^3T_4 \ldots {}^{k-1}T_k$$

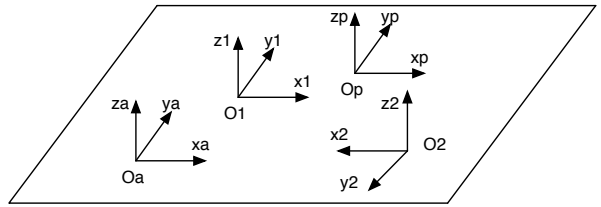
Invert transformations



$$T = \begin{pmatrix} A & P \\ 0 & 1 \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} A^T & \begin{matrix} -\underline{s}^T.P \\ -\underline{n}^T.P \\ -\underline{a}^T.P \end{matrix} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A^T & -A^T.P \\ 0 & 1 \end{pmatrix}$$

Exercice



$O_1^{R_a} = (10, 3, 0)^T \quad O_2^{R_a} = (20, 5, 0)^T \quad O_p^{R_1} = (5, 4, 2)^T$

Compute : ${}^{R_a}T_{R_1}$ and ${}^{R_a}T_{R_2}$

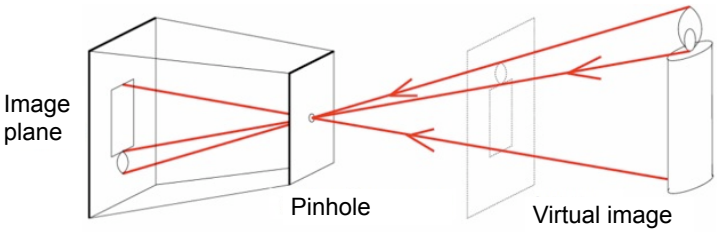
1 Before beginning
2 Camera models
1 pinhole model
2 other models
3 3D vision

Camera Models

Pinhole model

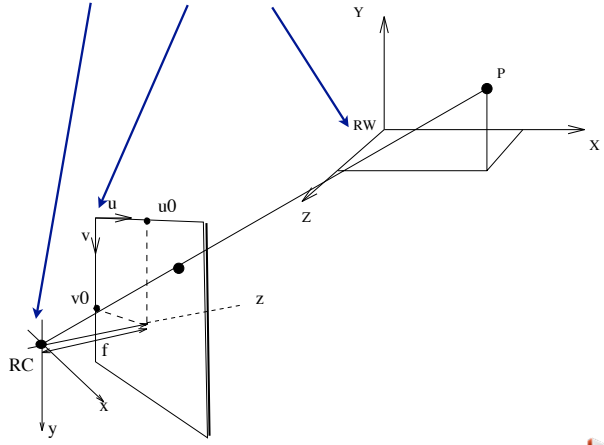
Camera Models

Pinhole model



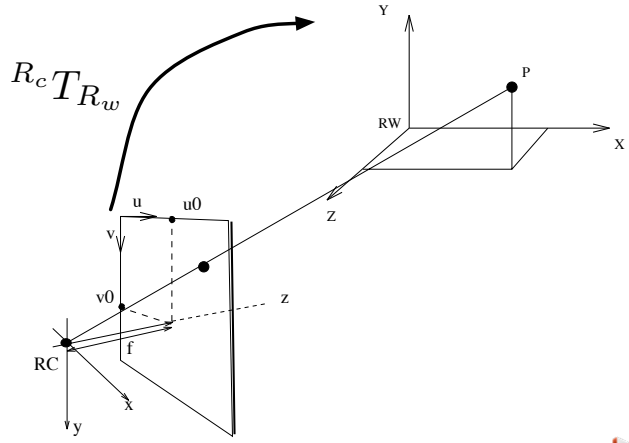
Camera Models

Pinhole model: 3 reference frames



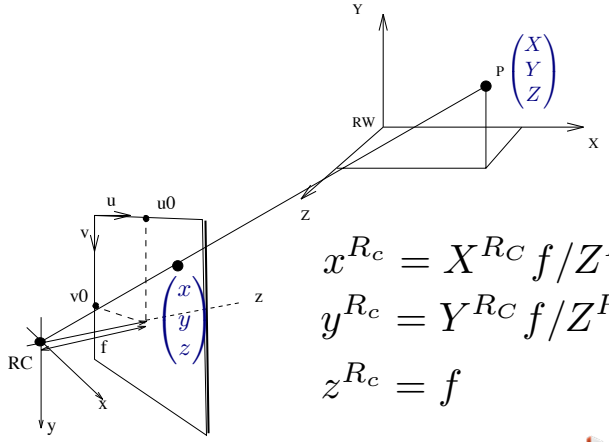
Camera Models

Pinhole model: extrinsic parameter matrix



Camera Models

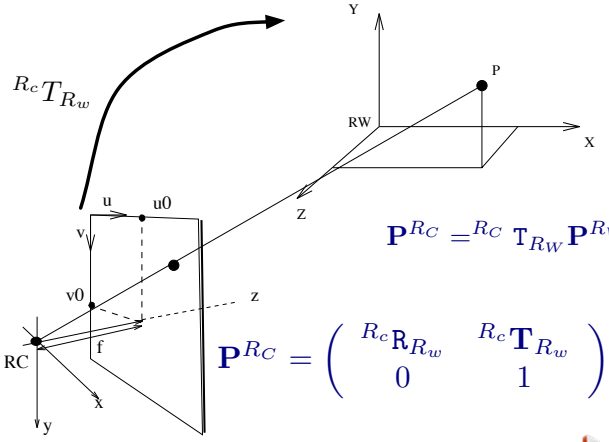
Pinhole model: perspective projection



$$x^{R_c} = X^{R_C} f / Z^{R_c}$$
$$y^{R_c} = Y^{R_C} f / Z^{R_c}$$
$$z^{R_c} = f$$

Camera Models

Pinhole model: extrinsic parameter matrix

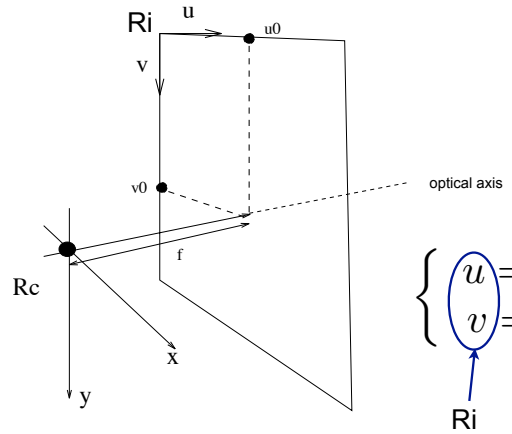


$$\mathbf{P}^{R_C} = {}^{R_C}\mathtt{T}_{R_W}\mathbf{P}^{R_W}$$

$$\mathbf{P}^{R_C} = \begin{pmatrix} {}^{R_c}\mathtt{R}_{R_w} & {}^{R_c}\mathbf{T}_{R_w} \\ 0 & 1 \end{pmatrix} \mathbf{P}^{R_W}$$

Camera Models

Pinhole model: image plane transformation



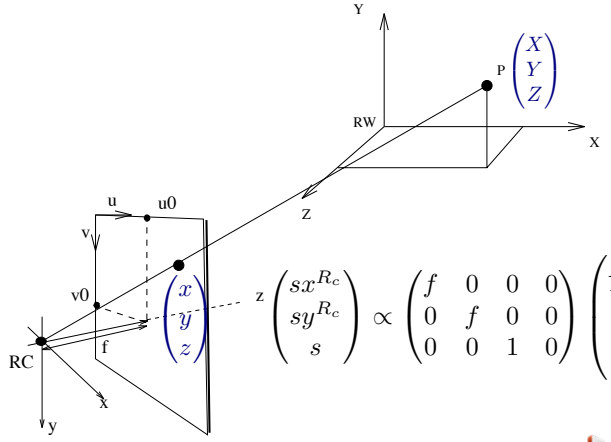
$$\begin{cases} u = x/dx + u_0 \\ v = y/dy + v_0 \end{cases}$$

Rc

Ri

Camera Models

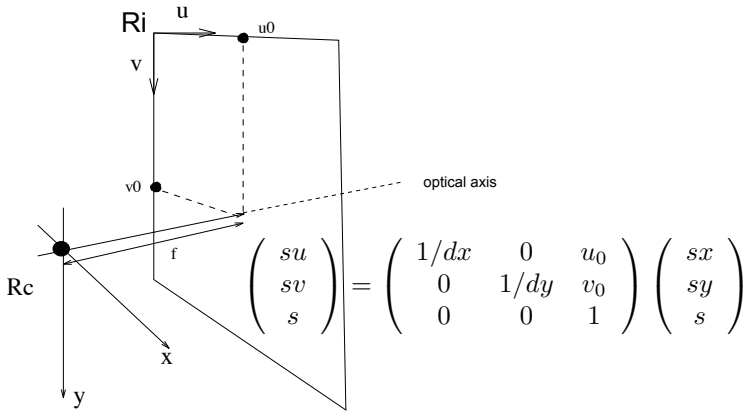
Pinhole model: perspective projection



$$\begin{pmatrix} sx^{R_c} \\ sy^{R_c} \\ s \end{pmatrix} \propto \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X^{R_c} \\ Y^{R_c} \\ Z^{R_c} \\ 1 \end{pmatrix}$$

Camera Models

Pinhole model: image plane transformation



$$\begin{pmatrix} su \\ sv \\ s \end{pmatrix} = \begin{pmatrix} 1/dx & 0 & u_0 \\ 0 & 1/dy & v_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} sx \\ sy \\ s \end{pmatrix}$$

Camera Models

Pinhole model: image plane transformation

$$\begin{cases} u = x/dx + u_0 \\ v = y/dy + v_0 \end{cases}$$

$$\begin{pmatrix} su \\ sv \\ s \end{pmatrix} = \begin{pmatrix} 1/dx & 0 & u_0 \\ 0 & 1/dy & v_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} sx \\ sy \\ s \end{pmatrix}$$

- (u_0,v_0) are pixel based coordinates into the image of the intersection between the optical axis and the image plane
- (dx,dy) are the x and y dimension of one pixel of the physical sensor.

Camera Models

Pinhole model: intrinsic parameter matrix

$$\mathtt{M}_{int} = \begin{pmatrix} fx & 0 & u_0 \\ 0 & fy & v_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left.\begin{matrix} f_x = f/dx \\ f_y = f/dy \end{matrix}\right\} \rightarrow (f_x/f_y = dy/dx)$$

pixels mm

Camera Models

Pinhole model: intrinsic parameter matrix

$$\begin{pmatrix} su \\ sv \\ s \end{pmatrix} = \begin{pmatrix} 1/dx & 0 & u_0 \\ 0 & 1/dy & v_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X^{R_c} \\ Y^{R_c} \\ Z^{R_c} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} su \\ sv \\ s \end{pmatrix} = \begin{pmatrix} f/dx & 0 & u_0 & 0 \\ 0 & f/dy & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X^{R_c} \\ Y^{R_c} \\ Z^{R_c} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} su \\ sv \\ s \end{pmatrix} = \begin{pmatrix} fx & 0 & u_0 & 0 \\ 0 & fy & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X^{R_c} \\ Y^{R_c} \\ Z^{R_c} \\ 1 \end{pmatrix}$$

Camera Models

Pinhole model: global projection matrix

$$\begin{pmatrix} su \\ sv \\ s \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{pmatrix} \begin{pmatrix} X^{R_W} \\ Y^{R_W} \\ Z^{R_W} \\ 1 \end{pmatrix}$$

$$\begin{cases} u = \dfrac{m_{11}X^{R_w} + m_{12}Y^{R_w} + m_{13}Z^{R_w} + m_{14}}{m_{31}X^{R_w} + m_{32}Y^{R_w} + m_{33}Z^{R_w} + m_{34}} \\ v = \dfrac{m_{21}X^{R_w} + m_{22}Y^{R_w} + m_{23}Z^{R_w} + m_{24}}{m_{31}X^{R_w} + m_{32}Y^{R_w} + m_{33}Z^{R_w} + m_{34}} \end{cases}$$

Camera Models

Pinhole model: global projection matrix

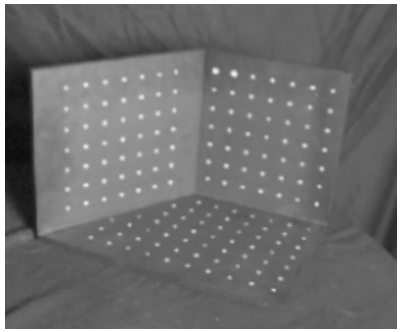
$$\begin{pmatrix} su \\ sv \\ s \end{pmatrix} = \mathtt{M}_{int}\ {}^{R_C}\mathtt{T}_{R_W} \begin{pmatrix} X^{R_W} \\ Y^{R_W} \\ Z^{R_W} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} su \\ sv \\ s \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{pmatrix} \begin{pmatrix} X^{R_W} \\ Y^{R_W} \\ Z^{R_W} \\ 1 \end{pmatrix}$$

P

Camera Models

Pinhole model: camera calibration



Given a set of 2D<->3P matching points, compute:

$$\mathtt{P} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{pmatrix}$$

Camera Models

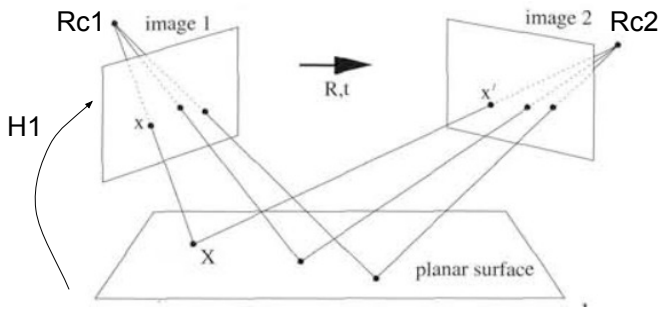
Pinhole model: camera calibration

1) built the linear system:

$$\begin{pmatrix} & & & & & & & \cdot & & & & \\ & & & & & & & \cdot & & & & \\ & & & & & & & \cdot & & & & \\ X_w^i & Y_w^i & Z_w^i & 1 & 0 & 0 & 0 & 0 & -u^iX_w^i & -u^iY_w^i & -u^iZ_w^i & -u^i \\ 0 & 0 & 0 & 0 & X_w^i & Y_w^i & Z_w^i & 1 & -v^iX_w^i & -v^iY_w^i & -v^iZ_w^i & -v^i \\ & & & & & & & \cdot & & & & \\ & & & & & & & \cdot & & & & \\ & & & & & & & \cdot & & & & \end{pmatrix} \begin{pmatrix} m_{11} \\ m_{12} \\ \cdot \\ \cdot \\ \cdot \\ m_{32} \\ m_{34} \end{pmatrix} = 0$$

2) Solve it with a least square linear optimization

Planar transformations



$$\mathbf{x} = (u, v, 1)^T$$
$$\mathbf{X} = (U, V, 1)^T$$

$$\mathbf{X} \propto \mathtt{H}_1 \mathbf{x}$$

1 Before beginning
2 Camera models
3 3D vision
1 Planar transformations
2 Multi-view 3D reconstruction

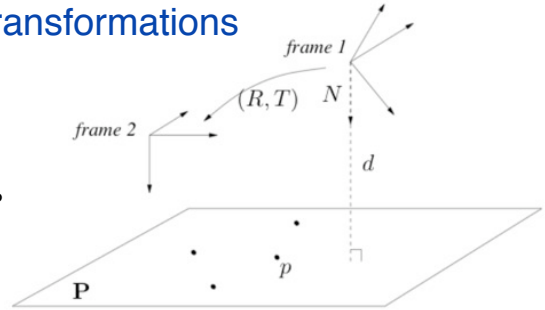
Planar transformations

$$\mathbf{X} \propto \mathtt{H}_1 \mathbf{x}$$

$$\mathbf{X} \propto \begin{pmatrix} h11 & h12 & h13 \\ h21 & h22 & h23 \\ h31 & h32 & h33 \end{pmatrix} \mathbf{x}$$

Planar transformations

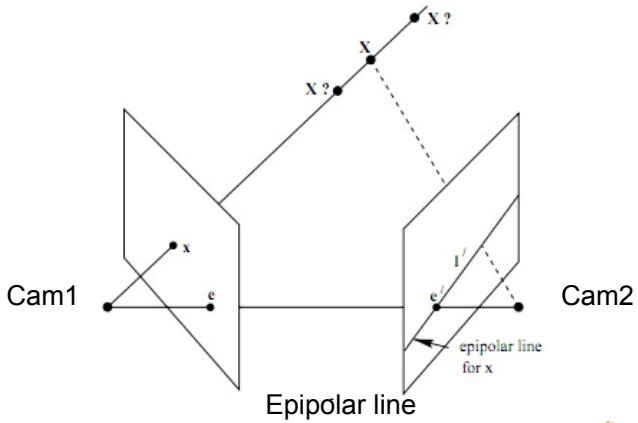
pc1 : point p in frame 1
pc2 : point p in frame 2
p : point in plane P



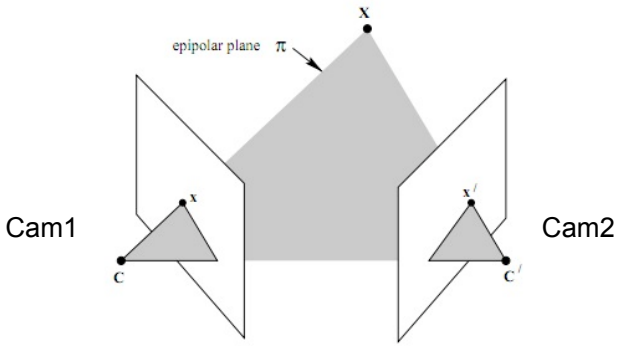
$$\mathbf{P}_2 = \mathtt{R}\mathbf{P}_1 + \mathbf{T}$$

$$\mathtt{H} = \mathtt{R} + \frac{1}{d}\mathbf{T}\mathbf{N}^T$$

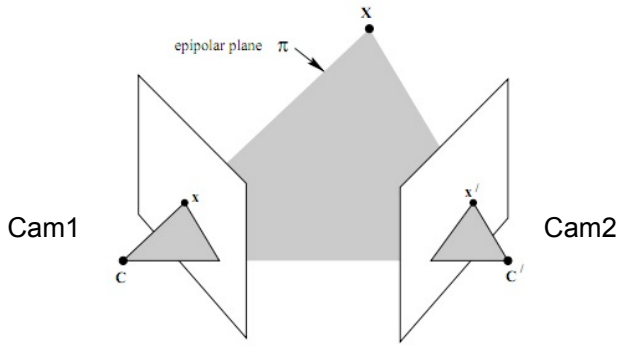
Multi-view 3D reconstruction (intro)



Multi-view 3D reconstruction (intro)



Fondamental Matrix



Fondamental Matrix : $\mathbf{x}^T \mathbf{F} \mathbf{x}' = 0$

With **F** a 3x3 matrix of rank 2.

Fondamental Matrix

The fondamental matrix is the key relation in structure from motion algorithms.
Further reading in:

```
@Book{Hartley2000,
    author = "Hartley, R.~I. and Zisserman, A.",
    title = "Multiple View Geometry in Computer
Vision",
    year = "2000",
    publisher = "Cambridge University Press, ISBN:
0521623049"
}
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Stereo

Two calibrated cameras see the same 3D rigid scene.

1) Write the linear system which gives the estimation of a 3D point from the projection of the point in the two calibrated cameras (in a least square way).

Homography

Two cameras see the same plane.

1) How many matches of image points are necessary to compute the Homography between the two images ?
2) Write the linear system which gives the estimation of the homography between the two cameras from matches image points (in a least square way).